Proceedings of the ASME 2014 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2014
AUGUST 17-20, 2014 BUFFALO, NEW YORK, USA

DETC2014-34638

The duality between the singularity of Bricard mechanism and the Singularity of Stewart platform

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Abstract

The duality (known also as symmetry) between serial chain manipulators and fully parallel mechanisms is well known in the literature. This paper takes this idea one step further, by introducing a systematic method that transforms mechanical systems into other and different mechanical systems so that the wrench screws in the original system gives rise to the relative twist screws in the second system. The mathematical foundation of this work relies on using the body-bar (BB) graph, a variant of graph representation widely used in mechanisms, possessing both the topology and geometry of the original system. From the dual graph of the latter it is possible to construct the dual system at a specific configuration. Relying on the equivalence between the dual systems, it is proved that if the screw system of a mechanism is at the singular position, so is that of its dual. This idea is demonstrated by showing the dual system of a Bricard mechanism, which is a 6/6 Stewart Platform in the singular position. The paper also shows that the cyclohexane molecule is dual to the 6/3 Stewart platform at the singular position, providing another perspective of the known mobility of this molecule. *Keywords: body bar graph, singularity, wrench screw, relative twist screw, cyclohexane molecule.*

1. Introduction

The duality, also called symmetry, between serial and parallel robots is widely known in the literature. One of the first works on this topic was done by Waldron and Hunt [1], who showed the dual relations between serial and parallel robots. Many other researchers followed this work, such as [2], and reported on the different properties between serial and parallel robots. Duality between systems by different graph representations was introduced [3], but this paper focuses on the duality between singular systems.

The paper first introduces how to represent mechanical systems by a graph representation, called Body-Bar graph. Once the graph possesses the topology of the system we add to it information about the screw axes (lines) of the system; the graph then reflects both the topology and geometry of the system. In a systematic way we then construct the dual graph, from which it is possible to construct the dual mechanical system. The work reported in the paper is not limited to parallel or serial robots, but can be a combination of them or other types.

In section 2 we introduce the Body-bar graph and indicate its relation to other known graph representations used in mechanisms. One of the contributions of this paper is that it introduces a way to add geometry information to the graph, a property that will be used while constructing the dual system. The paper introduces joints with one constraint in the original system thus in the dual graph there are two/five constraints for 2D and 3D, respectively. For such joints the geometry of the joints can be defined by one line **.**

In section 3 we establish the mathematical foundation of this work, where for each edge we associate a wrench screw or relative twist screw. While constructing the dual graphs, the wrench screws are transformed to relative twist screws and vice versa **.**

Section 4 introduces the main topic of the paper. It describes the duality singularity theorem that posits, if you construct the dual system of a given mechanical system through the method brought forth in the paper, you will derive another system at the singular position. We note that this concept needs further investigation since while constructing the dual system from the dual graph there are problems related to each graph that should be taken into consideration. For that reason, for now, the process is not yet fully deterministic.

In the end of section 4 we provide two examples: the first shows that the dual of Bricard mechanism is 6/6 Stewart platform, and the second shows the dual of cyclohexane molecule which is also 6/3 Stewart platform also at the singular position.

2. The weighted Body-bar(BB) graph of mechanical systems

There are several types of graph representations used to represent mechanical systems. In this paper we first represent it by a body-bar graph, for short we write BB, which is somehow similar to the graph representation used by Freudenstein,[4], Tsai [5] and others, and then for sake of simplicity we present it as a weighted graph.

2.1 The topology of the weighted Body-bar graph

The graph G=(VB,E) is defined to be a body-bar graph if the vertices correspond to the bodies/links and the edges between two vertices stand for a joint connecting between the two bodies and the number of edges is equal to the number of constraints imposed by that joint. An edge can stand for a distance constraint or a rotation constraint depending on the type of the joint.

For instance, in the BB graph in Figure 1.b there are two edges between bodies 1 and 2 since the kinematic pair is of type revolute joint. Both constraints account for translational constraints. There is a gear pair, which is a higher pair, between bodies 2 and 3 thus only one edge appears between vertices 2 and 3. The BB is similar to the graph representation used in the mechanisms literature [5] as shown in Figure 1.c, where higher pairs are represented as solid bold lines. For sake the of constructing the dual graph, when there are several edges between two edges, termed parallel edges, we replace them with a weighted edge, i.e., a number (designated with an underlined italic number) is associated to each edge indicating the number of edges that are parallel. In case there is only one edge between the two vertices, no number is associated as shown in Figure 1.d.

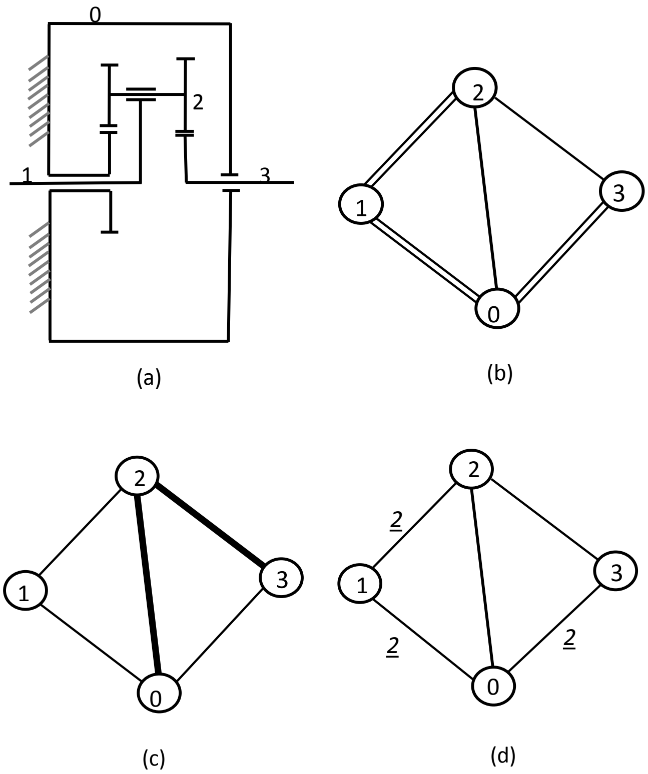


Figure 1. A gear train (a), its corresponding BB graph (b), the graph representation according to Tsai's book (c) and the weighted BB graph (d).

The main advantage of the BB graph is that it enables to represent any mechanism with any type of joint. Only if there are so-called multiple joints the graph is not unique. In this case it is possible to use the extension of this graph, termed mixed graph [6], but this is not the subject of this paper and multiple joints are excluded w.l.o.g**.**

Once the graph possesses the topology of the mechanical system we add to it information about the instantaneous geometry.

2.2 Adding the geometry of the mechanical system to the graph

In this section we add information to the graph about the locations/lines defining the kinematic pairs. In the example of gear trains we augment the locations of all the turning pairs. In the case of gear trains we have to provide the locations in space of the gear pairs, designated by the letter 'g' and the number of the two wheels, while lower letters stand for the locations of the lower kinematic pairs. For instance, in Figure 2.b the weighted BB graph possesses all the needed topology and geometry information so its dual system can be constructed.

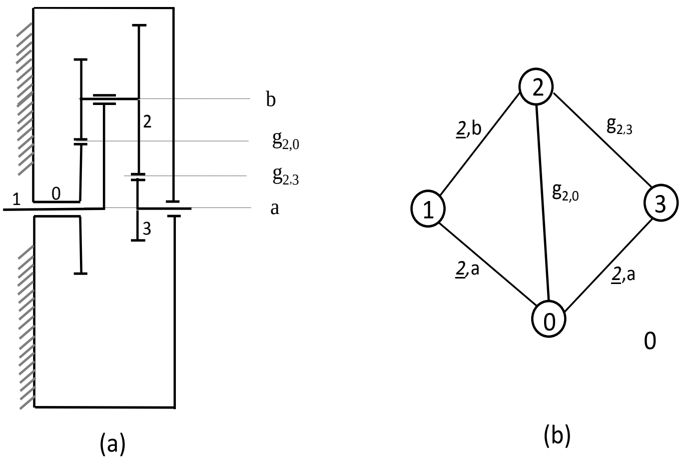


Figure 2. The gear train (a) and its weighted BB graph (b) augmented with the geometry information.

Once the graph possesses all the information of the mechanical system, including both topology and geometry, we have the needed information to construct its dual graph. First we will find the topology of the dual graph and then we will transfer the geometry to the dual system.

3. Constructing the dual mechanical system

In this section we introduce a method that enables to transform mechanical systems into their corresponding dual systems through the BB graph. The aim of this approach is that this process will be fully algorithmically, but for now the process of constructing the dual system from the dual graph needs taking into consideration problems specific for dual system thus this process is not yet fully deterministic.

3.1 Constructing the topology of the dual weighted BB graph

Given the wBB graph GB=(VB,E) we construct the dual graph G*B=(V*,E*) as follows:

for each face (a circuit with no inner edges) we associate a vertex in G*. If two faces Fi and Fj are adjacent, (there is a common edge between the two faces), we add an edge in the dual graph between the corresponding two vertices, i.e., $(v_i^*,v_j^*)\in E^*$. Note, the number of edges in the original graph is equal to the number of edges in its dual graph, i.e., |E|=|E*|, thus for each edge in the original wBB there exists a dual edge in the wBB*, called the corresponding dual edge. Since the edges correspond to joints, joints are transformed into another type of joint in the dual system. The weight (number of parallel edges) of the corresponding dual edge of 'e', i.e., w(e*) is determined as follows:

$$w(e^*) = d(B) - w(e), \qquad (1)$$

where d(B) is the DOF of an unconstrained rigid body, i.e. for planar mechanisms d(B)=3, and for spatial mechanisms d(B)=6.

It follows from equation 1 that in general lower kinematic pairs are transformed into higher pairs. Also a higher pair is possibly transformed to a lower pair. For instance, a distance constraint (a bar between two bodies) in 3D, which defines one constraint, is transformed to a hinge joint, which imposes five constraints.

In Figure 3 we construct the dual wBB of the gear train given in Figure 2.a. There are three faces therefore in the dual graph there are three vertices denoted by: 0, I and II. The weight of the dual edge is computed according to equation 1 as follows: w(e*) = 3 – w(e). For example, edge (0,2) is not weighted, i.e., w(0,2)=1 thus the weight of its dual w*(I,II) = 3 – 1=2, as appears in Figure 3.b.

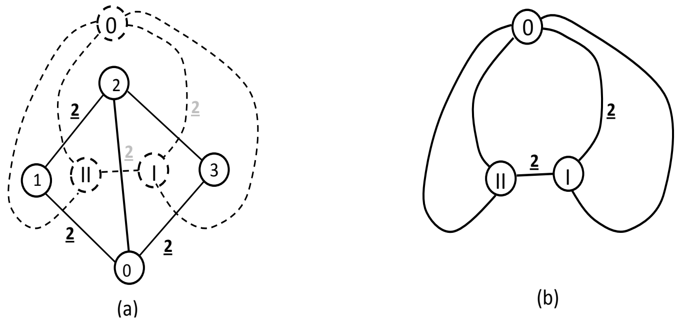


Figure 3. Constructing the dual graph of the gear train given in Figure 2.a

(a) The wBB (solid lines) and its dual graph – wBB* (dashed lines). (b) The dual wBB* alone.

3.2 Adding the geometry to the dual graph

In kinematics we are more interested in the locations of the joints than the configuration of the bodies since we are interested in the relative twists between the two connected bodies. Therefore, at this step we add the necessary information regarding the geometry of the joints to the corresponding edges. For example, for revolute joints we add the geometry of the axis of the joints, for planar gear pairs we add their locations, and so forth**.**

As was introduced in the previous section, the edges correspond to the joints and for each edge there is a corresponding dual edge representing the dual joint. Thus, the geometry of the dual joint is transformed from the original graph through the edges as stated in the following property**:**

Property 1: Every joint in the original system represented by edge 'e' with geometry g(e) is transformed into its dual joint corresponding to edge 'e*' by applying the following equality**:**

g(e) = g(e*) (2)

meaning that the geometries defining the original joint and its dual joint are the same-.

An example of transforming the geometry of the joints from the original graph to its dual graph is given in Figure 4. Firstly, the geometry of the joints is associated to the edges of the original BB as shown in Figure 4.a. Each edge in the dual graph corresponds to an edge in the original graph as shown in Figure 4b, thus the geometry given in Figure 4.c can be transformed to the edges of the dual graph as shown in Figure 4.d.

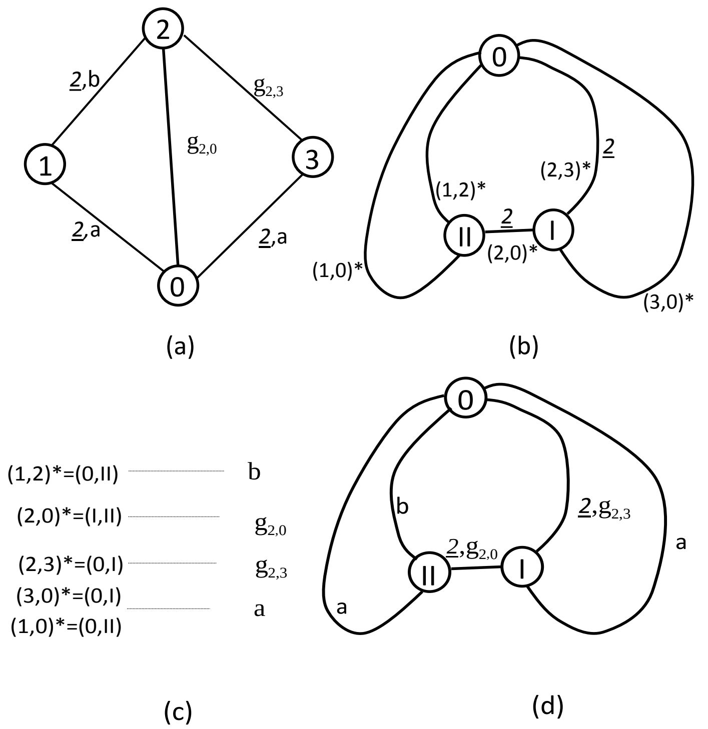


Figure 4. Transforming the geometry information from the wBB graph of the gear train (Figure 2.b) to its dual graph .

a) The original wBB with the geometry of the joints associated to its edges. b) The dual graph. c) The geometry of the joints of the gear train . d) The resultant dual graph of the gear train with the geometry of the joints associated to its edges.

3.3 Constructing the dual mechanical system from the dual graph

Now that the dual graph also possesses the geometry information, it is possible to construct from it the dual mechanical system following the construction rule given below:

Construction rule: every joint remains in its place but its type is changed according to equation 1 and the two bodies that it connects are different and defined by the dual graph.

For instance, let us follow the process of constructing the joint (0,II) in the dual graph corresponding to edge (1,2) in the original graph. The edge (1,2) corresponds to a revolute joint (i.e., imposes two constraints) at level b. Following the construction rule the dual joint is also located at level b and imposes one distance constraint but now connects between body I and the ground as shown in Figure 5.a1. In the case of a higher pair joint, such as the gear pair g2,0, the dual joint is a lower pair – revolute joint as shown in Figure 5.b1. The same process is applied for each edge of the dual graph until the dual system is constructed, as shown in Figure 5.c1.

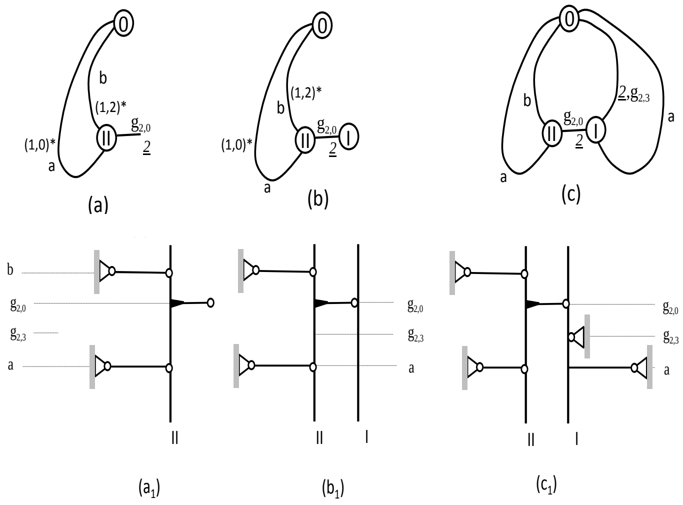
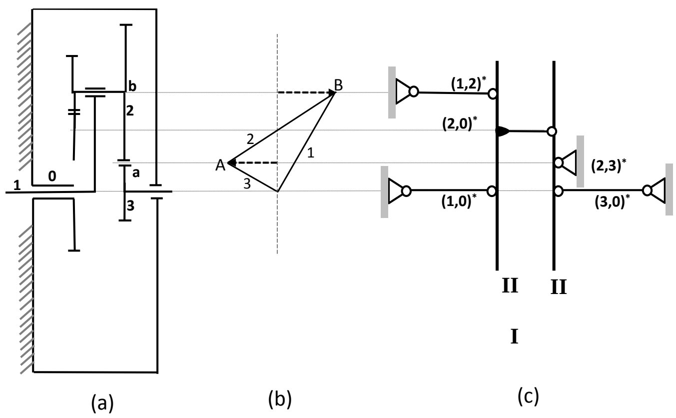
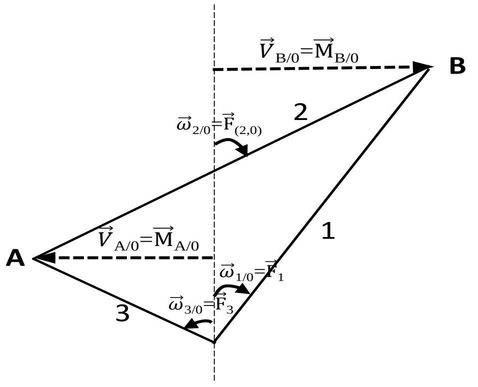


Figure 5. Example of constructing the dual system of the gear train (Figure 2.a) from its dual graph following the construction rule.

In Figure 6 we can see the equivalence between the angular/linear velocity diagram of the original gear train, Figure 6.a, and the force/moment diagram of the dual static system, Figure 6.d. In 3D this duality will provide us equivalence between twist screws and wrench screws as described in section 3.4.3.

It should be noted that the similar behavior between the original and the dual system was not obtained by writing any equation, but by simply constructing the dual system from the dual graph. The reason and the mathematical proof underlying this similarity is provided in the next section.





(d)

Figure 6. The equivalence between the angular/linear velocities of the gear train and the forces/moments of the dual system.

a)The original gear train. b) The velocity diagram of the gear train. c) The dual mechanical system. d) The unified diagram for both velocities and forces of the original and dual systems, respectively.

3.4 The mathematical proof underlying the equivalency between the original and dual systems

In section 3.1, how to construct the dual graph from the original graph was explained. In this section we explore the relation between circuits and their duals, cut-sets, defined below. For this we must first introduce a few definitions from graph theory: a path between two vertices is a set of edges that starts at one vertex and ends at the other. If there is a path between any two vertices the graph is called a connected graph. A circuit is a closed path, i.e., a path that starts and ends at the same vertex. A cut-set is a minimal set of edges whose removal disconnects the graph. For example, the set of edges {1,2,4} in Figure 7.a is a cut-set because after removing them the graph is disconnected (since there is no path ,for example, between vertex B to vertex A). On the other hand, the set {2,5} is not a cut-set since after removing the two edges the remaining graph is still connected.

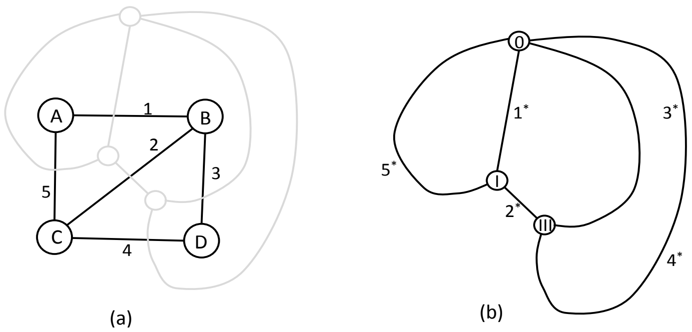


Figure 7. Example of a graph and its dual**.**

a) The original graph (solid lines) and its dual graph – (gray lines). b) The dual graph alone.

Hereinafter is the main theorem from graph theory that amalgamates cut-sets in the original graph and circuits in the dual graph, and vice versa, as follows:

Theorem [7]: Let G=(V,E) be a connected graph, E' is a subset of edges and E'* is the corresponding subset of edges in the dual graph G*=(V*,E ,(*

E' is a cut-set in G IFF E'* is a circuit in G* and vice versa.

For example, the set of edges {1,2,5} is a circuit in the original graph, Figure 7.a, while the corresponding edges {1*,2*,5*} in the dual graph, Figure 7.b, is a cut-set .

Based on this relation between cut-sets and circuits we can reveal the relation between the kinematic and static equations of two dual systems as given in section 3.4.3.

For each edge 'e', representing a joint addressing the forces and moments in that edge we associate a wrench screw of edge 'e', designated $w(e). When we relate to the relative angular and linear velocities between the two bodies connected by the joint represented by edge e we then associate a twist screw of edge e, designated by $t(e).However, many such 'exceptional' systems are used in practice, which calls for an alternative to the CKG formula. Such an alternative is the use of a combinatorial method, called the Pebble game algorithm that was developed in the context of rigidity theory. This method requires an appropriate graph representation of the kinematics, namely body-bar and bar-joint graphs, as will be explained in this paper. Beside their relevance for the pebble game algorithm these graph representations give rise to the so-called rigidity matrix -the central object in rigidity theory. Currently the rigidity theory approach is limited in that it only allows for spherical joints. In this paper the necessary constraints for prismatic joints are derived that enable extension of the body-bar approach to general planar systems. To the authors' knowledge the mechanism analysis using body-bar graphs has not been reported so far.

3.4.1 The equilibrium of the wrench screws in the cut-sets of the BB graph

As was mentioned above, the vertices of the BB graph represent bodies/links and the edges represent the joints. From the viewpoint of statics, each weighted edge stands for the forces and moments that are transmitted by the corresponding joint to the two bodies that it connects .

For satisfying the force and moment equilibrium in the mechanical system represented by the BB graph, the sum of all the wrench screws in all the cut-sets should be equal to zero. In mathematical form, it can be written that for any cut-set Qk the summation of all the wrench screws , $w(e), of edges 'e' belonging to the cut-set Qk is equal to zero as follows:

$$\sum_{\forall e \in Q_k} \$^{w}(e) = 0 \qquad (3)$$

Equation 3 can be thought of as the extension of the Kirchhoff's Current Law (KCL) [8] into six dimensions, where each dimension relates to one of the six components of the wrench screws in the cut-set.

3.4.2 The constraints of the twist screws in the circuits of the BB graph

In kinematics we ascribe for each edge of the BB graph the difference between the twist screws of two bodies corresponding to the two end vertices of the edge. Namely, for each edge corresponding to a joint, the twist screw $t(e) associated to edge 'e' is the relative twist screw of the two end vertices of that edge **.**

For each circuit, say Cm, the sum of all the twist screws belonging to this circuit should be equal to zero, as follows:

$$\sum_{\forall e \in C_m} \$^{t}(e) = 0 \qquad (4)$$

Equation 4 can be thought of as the extension of the Kirchhoff's Voltage Law (KVL) into six dimensions, where each dimension relates to one of the six components of the twist screws in a circuit**.**

A special case of equation 4 in 2D is the "contour equation" [9], as is used in following section **.**

In the next section we introduce the relation between the twist screws and wrench screws in the dual graphs and show when they can be the same.

3.4.3 The duality relation between the wrench screw and the relative twist screw in the dual BB graphs

In this section we introduce the main theorem underlying this paper. We show that when we construct the dual graph, for any edge in a cut-set the wrench screw becomes a twist screw in the dual edge now belonging to the dual circuit, and vice versa, as stated in the following theorem**.**

Duality Theorem: the duality theorem between twist and wrench screws**:**

Let G be a BB graph of a mechanical system M and G* its dual BB graph constructed as explained in section 3.3 and the geometry of edge 'e' and its dual edge 'e*' is the same, i.e., g(e)=g(e*), then**:**

Let Q be a cut-set in G and C* its dual circuit in G*, then for any edge 'e' belonging to Q and its corresponding dual edge 'e*' in C* the wrench screw of edge 'e' is equal to the twist screw of edge 'e*' , up to multiplication with a scalar □, as follows:

$$e \in Q,\ e^* \in C^* : \$^{w}(e) = \gamma \$^{t}(e^*) \qquad (5)$$

where the interpretation of wrench screw is $w=(F,T), with force F and torque T, and of relative twist screw is $t=(□r,v), with relative angular velocity □r and linear velocity v.

Proof: We choose a cut-set Q in G, thus according to equation (3) the sum of all the wrench screws in that cut-set is equal to zero. Since cut-set Q is a circuit C* in the dual graph G*, thus for each edge e belonging to Q there is a corresponding edge 'e*' belonging to C*. According to the condition of the duality theorem, the geometry of each edge is identical to the geometry of the corresponding dual edge the set of equations for the wrench screws in cut-set Q becomes the same set of equations for the twist screws in the dual circuit C*. It follows that each wrench screw in edge 'e' is equal to the twist screw in the corresponding dual edge 'e*', up to multiplication with a scalar for all the edges**.**

It should be noted that we are discussing the values of the screws but their unit of measurement is different, for twist screws it is velocities and for wrenches it is forces and moments. Q.E.D**.**

For the sake of clarity, all the joints of the 3D mechanical systems have one (bar) or five (hinge joint) constraints, thus the geometry defining the joints is just a line**.**

To clarify the duality theorem we provide a detailed example in 2D where all the kinematic equations in the original graph turn out to be the static equations of the dual graph. Let us go back to the gear train in Figure 2.a whose dual static system was constructed through the dual graph. Let us start with the circuit {0,3,2} in Figure 8.a, which is the BB graph of the gear train, Figure 8.a1. The sum of the relative angular and linear velocities in this circuit, according to the contour method [8], should be equal to zero, as follows:

$$\vec{\omega}_{\frac{3}{0}} + \vec{\omega}_{\frac{2}{3}} + \vec{\omega}_{\frac{0}{2}} = 0 \qquad (6)$$

$$\vec{\omega}_{\frac{3}{0}} \times \vec{r}_1 + \vec{\omega}_{\frac{2}{3}} \times \vec{r}_2 + \vec{\omega}_{\frac{0}{2}} \times \vec{r}_{21} = 0 \qquad (7)$$

This circuit corresponds to a cut-set in the dual graph, Figure 8.b, and applying the force and moment equilibrium for this cut-set yields:

$$\vec{F}_{(3,0)*} + \vec{F}_{(2,3)*} + \vec{F}_{(0,2)*} = 0 \qquad (8)$$

$$\vec{F}_{(3,0)*} \times \vec{r}_1 + \vec{F}_{(2,3)*} \times \vec{r}_2 + \vec{F}_{(0,2)*} \times \vec{r}_{21} = 0 \qquad (9)$$

Hence the screw coordinates of the original and the dual mechanism are indeed identical.

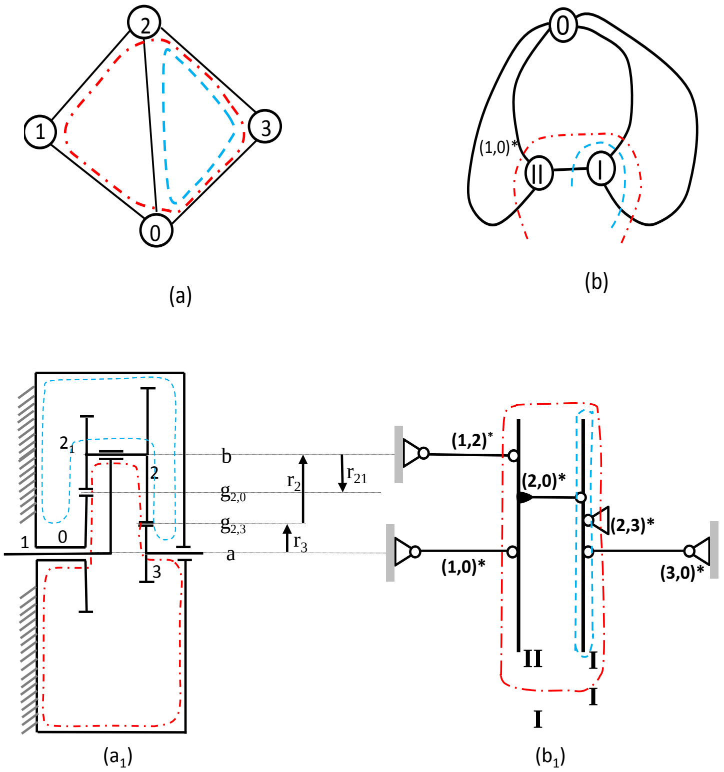


Figure 8. The equivalence between the gear train (a1) and its dual system (b1.(

a) The BB graph of the gear train (a1)and two of its circuits. b) The BB graph of the static system (b1) and two of its cut-sets.

4. The duality of singularities of mechanical systems through the BB graph

In section 3 we introduced the process of constructing the dual of mechanical systems. We first constructed the BB graph of the original system including the geometry of the joints. Then, we constructed the dual graph of the BB and transformed the geometry of the joints to the dual joints. At last, from the dual graph we constructed the second mechanical system and according to the duality theorem (section 3.4.3), the twist screw of an edge belonging to a circuit is the same as the wrench screw of the dual edge belonging to the dual of the circuit, this time a cut-set**.**

In this section we take this idea one step further and start with a mechanical system at the singular position, therefore its dual mechanical system will be also at the singular position, as stated in the following theorem.

The duality singularity theorem: Let M1 be a mechanical system and G its BB graph possessing the geometry of the joints. Let G* be the dual graph of G by transforming the geometry of the joints to their corresponding dual joints. From G* we construct a second mechanical system, let us call it M2, then**:**

M1 is at the singular configuration iff M2 is at the singular position**.**

Proof: the proof follows the proof of the duality theorem (section 3.4.3). Since all the twist screws in a circuit become wrench screws in the dual cut-set, and vice versa, the governing equations of the original graph G are identical to the equations of the dual graph, so their BB matrices are the same. Thus, if the determinant of the BB matrix of G is equal to zero it follows that the determinant of the BB matrix of G* is also equal to zero**.**

Hereinafter we provide some examples of mechanical systems at the singular position and their dual singular mechanical systems, starting with 2D.

4.1 Examples of duality singularity of 2D mechanical systems

Let us start with a 2D Triad in a generic configuration, i.e., not in the singular position. The 2D triad consists of two bodies, I and II, and three joints connecting between them, bars 1,2 and 3, each imposing one constraint (higher pair) as shown in Figure 9.a. One can think of the 2D triad as a 2D variant of Stewart Platform. There are three faces in the BB graph therefore in the dual BB graph there are three vertices in a circuit and between any two vertices there are two edges corresponding to lower pairs. The dual mechanical system consists of three links each two are connected by a revolute joint as shown in Figure 9.c. Note, the revolute joints are the dual joints of the three bars thus each axis of the revolute joint is in the direction of its corresponding bar. It turns out that the geometry of the joints of the original system is the same as the geometry of the dual joints. Note, the original system and its dual are static systems, i.e., immobile systems.

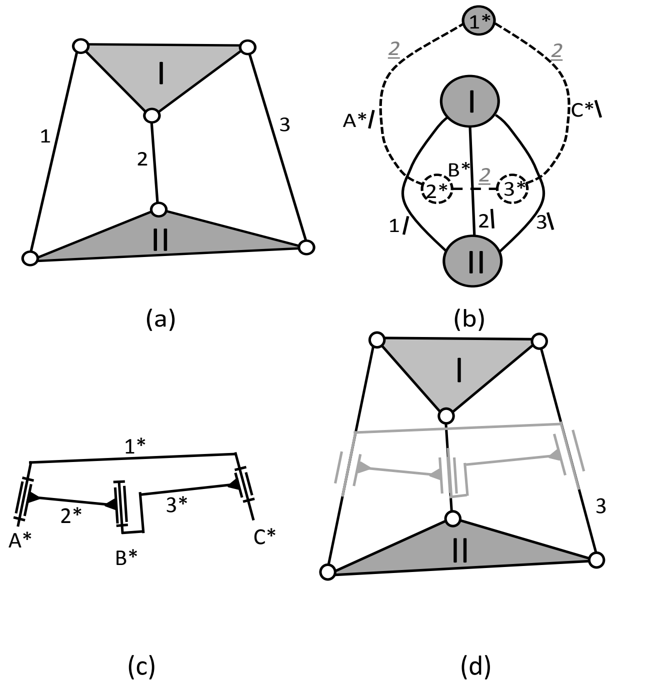


Figure 9. The dual system of a 2D triad in a generic position. a) The 2D triad. b) The BB graph and its dual. c) The dual mechanical system. d) The 2D triad and the superimposed dual system (gray).

If the 2D triad is in the singular position, then according to the singularity theorem its dual system should also be in the singular position as shown in Figure 10.

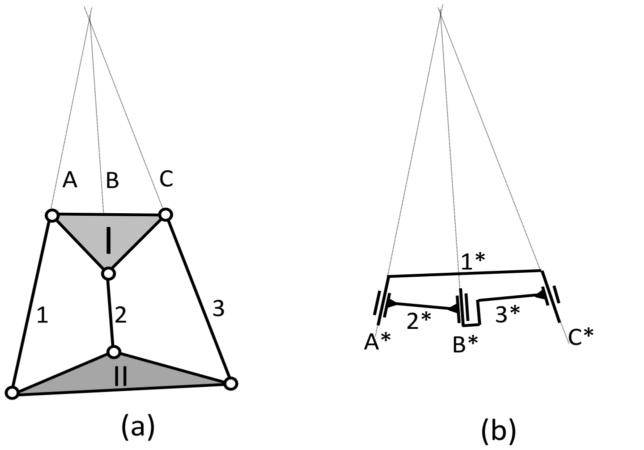


Figure 10. The 2D Triad (a) and its dual system (b) both in the singular configurations

It should be noted that the duality singularity theorem assures that both systems, the original and its dual, should be in their singular positions but can't distinguish the type of the singularity. For example, static systems in a singular position can have an infinitesimal motion or a finite motion, and both are types of singularity. Therefore, there can be a configuration of the 2D Triad possessing infinitesimal motion while its dual system has a finite motion, as shown in Figure 11.

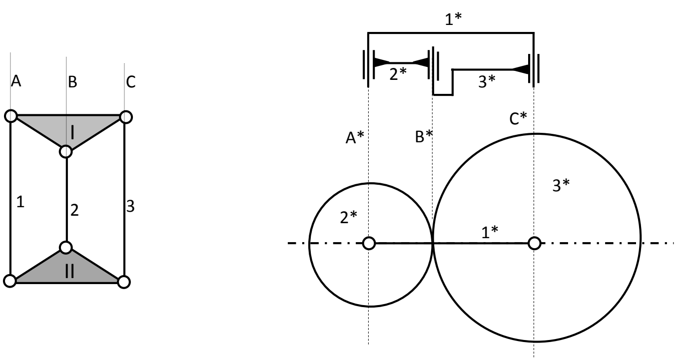


Figure 11. A 2D Triad in a singular position with an infinitesimal motion (a) while its dual system is in a singular position with a finite motion, two wheels rolling without sliding (b).

4.2 Examples of duality singularity of 3D mechanical systems

The relation between the singular position of the original mechanical system and its dual in 3D is the same as in 2D, i.e., the original system can have an infinitesimal motion while its dual can have a finite motion.

4.2.1 The singularity duality between 6/6 Stewart platform and Bricard mechanism

Stewart platform is represented by the BB with two vertices and six joints (the six legs) each imposing one constraint (Figure 12.a), whereas the dual joints impose five constraints, i.e., hinge joints. Therefore, in the dual system, Bricard mechanism, there are six hinge joints (Figure 12.b). In Figure 12 we used the known dimensions of the Bricard mechanism, thus according to the duality singularity theorem it is proved that the Stewart platform is in the singular position.

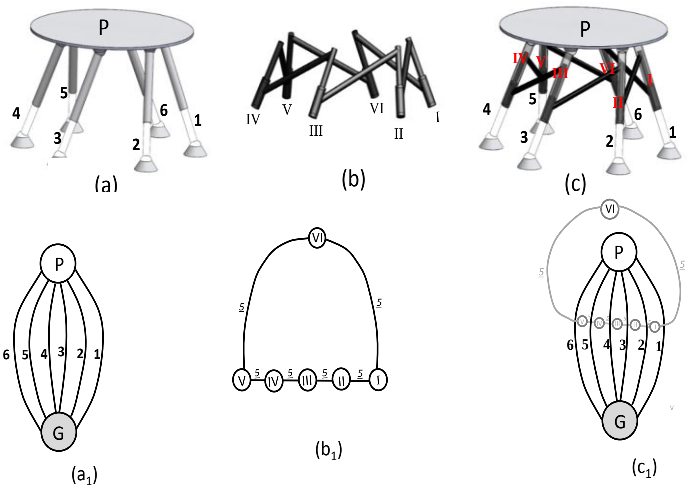


Figure 12. The dual singularity between 6/6 Stewart platform and Bricard mechanism**.**

a,a1) The Stewart platform and its BB graph. b,b1) Bricard mechanism and its BB graph. c,c1) The two dual systems and their BB graphs.

4.2.2 The singularity duality between 6/3 Stewart platform and Cyclohexane molecule

In this case we start from the cyclohexane molecule (Figure 13.b) known to a finite motion, due to its singularity. Once we have the sizes of the cyclohexane molecule, we are able to construct its BB graph (Figure 13.b1), its dual BB graph (Figure 13.a1) and from it we derive the 6/3 Stewart Platform (Figure 13.a).

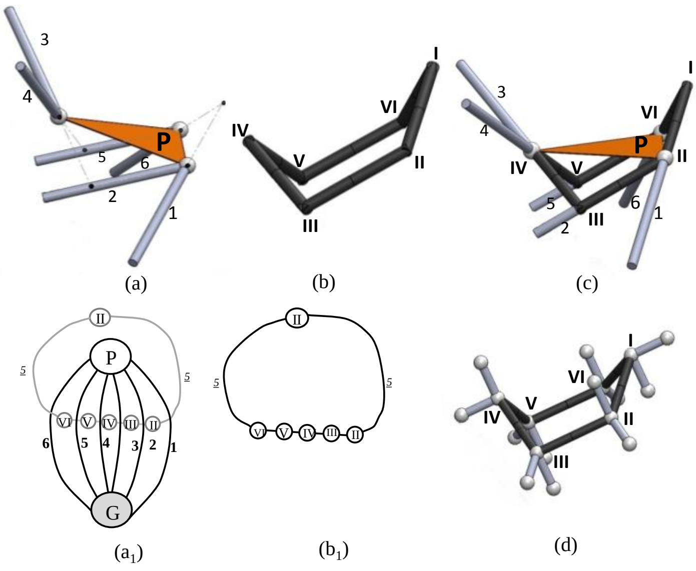


Figure 13. The dual singularity between 6/3 Stewart platform and Cyclohexane molecule.

a,a1) The 6/3 Stewart platform and its BB graph. b,b1) The structure of the Cyclohexane and its BB. c) The 6/3 Stewart platform and the superimposed cyclohexane structure. d) The cyclohexane molecule.

5. Conclusions and Further Research

A method for transforming mechanical systems to their corresponding dual systems through a topological graph, BB graph, has been introduced in the paper. By adding essential information about the geometry of the original system, such as the locations of the axes, it enables construction of the dual system in a specific geometry. In the paper we showed mechanical systems with one constraint or two/five constraint in 2D/3D constraints, since they can be defined through one line. We intend to present all the types of joints in the forthcoming paper .

There are several topics that are expected to follow from the reported work. One is to make the process of transforming the original system to its dual deterministic, opening the possibility of computerizing this approach. Another direction is extending the types of joints for which it will be possible to construct dual joints.

This work is expected to provide additional to the works reported about duality [9,10] insight on the singularity topic, this time through its dual systems.

ACKNOWLEGEMENTS

The first and second authors would like to thank Israel Science Foundation for their financial support, grant No. 1668/11.

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